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Algebraic brane dynamics on SU(2): excitation spectra*

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Abstract: We analyze the dynamics of $D2$-branes on SU(2) within a recently proposed matrix model, which works for finite radius of SU(2). The spectrum of single-brane excitations turns out to be free of tachyonic modes. It is similar to the spectrum found using DBI and CFT calculations, however the triplet of rotational zero modes is missing. This is attributed to a naive treatment of the quantum symmetries of the model. The mass of the lightest states connecting two different branes is also calculated, and found to be proportional to the arc length for small angles.

Keywords: Brane Dynamics in Gauge Theories, Quantum Groups, Matrix Models.

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Contents

1. Introduction

In recent years the structure of D-branes in a WZW models has attracted much attention. The background of the compact Lie groups G is known to carry a nontrivial (NSNS) B field which is not closed. It has been shown, using CFT [1, 2] and DBI (Dirac-Born-Infeld) [3] descriptions, that stable branes can wrap certain conjugacy classes in the group manifold. The problem of the brane description has also been attacked from a different perspective, by the matrix model [4] of D0-branes. The latter approach was supported by CFT calculations [5] and led to a beautiful picture where, in a special limit, the macroscopic branes could be viewed as a bound state of $D0$-branes. Finally, the spectra of branes in WZW models were calculated using K-theory [6]–[8].

These various approaches focus on different aspects of D-brane physics, and have different range of applicability. For example, the DBI approach is valid only in the large k limit, while the standard matrix model can handle only a certain subset of "small" branes. CFT is the most complete description, but it lacks the physical picture of the brane, and involves too many degrees of freedom.

Attempting to reconcile these various approaches, we proposed in [9] a matrix description of D - branes on SU(2) based on quantum symmetries. This led to a quantum algebra of (noncommutative) matrices, which reproduces all static properties of all stable

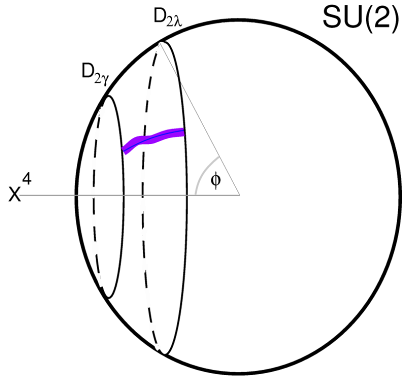

Figure 1: D2-branes on SU(2), with string. The angle is $\phi = \frac{(2\lambda+1)\pi}{k+2}$.

D-branes on SU(2), including finite size effects. This was extended in [10] to cover all untwisted branes for higher groups such as SU(N). In these papers, we were concerned only with static properties of the branes, ignoring the brane dynamics. This is the problem we want to address here, attempting to describe the dynamics of the branes in terms of gauge theories based on our quantum algebra.

In the present paper we calculate the spectra of excitations of branes within this matrix model. Because the paper is quite technical and the results are easy to state we present them here. The resulting excitation spectrum for a single brane $D_{2\lambda}$, $\lambda \in \{0, \ldots, k/2\}$ is (in the large k approximation)

$$\begin{aligned} m^2 &= \alpha^2\, j^2\,, \qquad & j &= 1, 2, \ldots, 2\lambda \\ m^2 &= \alpha^2\, (j+1)^2\,, \qquad & j &= 0, 1, \ldots, 2\lambda\,, \end{aligned} \tag{1.1}$$

where $\alpha \sim r/k$ and r is the radius of SU(2) (which should be $r \sim \sqrt{k}$). The spectrum (1.1) is very close to the known results [3]. In particular there is no tachyon, which shows that the model is quite reasonable as the branes are stable. Unfortunately, (1.1) does not contain the massless modes corresponding to the freedom of rotating branes inside S^3, which it should. We shall discuss this point at length in the paper.

Furthermore, the lightest states connecting two different (parallel) branes $D_{2\lambda}$ and $D_{2\gamma}$ have masses

$$m^2 = 4r^2 \sin^2 \frac{(\lambda - \gamma)\pi}{k+2}\,. \tag{1.2}$$

This formula gives $4r^2(\frac{(\lambda-\gamma)\pi}{k+2})^2$ in the large k approximation, which is the arc length of the string stretched between the two branes. This is in perfect agreement with stringy intuition and results obtained by other means [8]. For finite k and large angles, the result deviates from this simple geometrical interpretation, and it would be quite interesting to verify this. The scenario is sketched in figure 1.

We should point out that the lack of rotational zero-modes in this model is presumably due to an oversimplification in the way we treat the model. In order to keep things as simple as possible, we have considered our action as an ordinary, classical matrix model. We will discuss later that this is inconsistent with the (thus far formal) quantum group symmetries, so that they do not guarantee zero modes as they should. Hence this work should be seen as a first step towards a fully consistent treatment. Nevertheless, we believe that the models are very interesting as they are, being perhaps the first matrix models for branes on a curved background.

The detailed calculations leading to the above results are presented in section 4.1 where we analyze the spectra of single branes, and in section 4.2 where we consider excitations of two parallel branes. These two sections form the heart of the paper. They are preceded by an introduction of the basic variables and equations we are going to use. We close the paper with a discussion of the results, and several appendices containing technicalities.

2. Branes in the SU(2) WZW model

In this section we recall the necessary facts about D-branes in the SU(2) WZW model at level k. It is based on results obtained in several papers [1, 11, 3, 7], and contains no new material.

We know that there is only a discrete set of stable D-branes on $G = \mathrm{SU}(2)$ (up to global rotations), one for each integral weight[1] $2\lambda \in P_k^+ = \{0, 1, \dots k\}$. They are given by the conjugacy class $D_{2\lambda} = \{g^{-1} t_{2\lambda}\, g\}$ of the SU(2) element

$$t_{2\lambda} = \exp\left(2\pi i \frac{H_{2\lambda} + H_\rho}{k+2}\right) = \begin{pmatrix} q^{2\lambda+1} & 0 \\ 0 & q^{-(2\lambda+1)} \end{pmatrix}, \tag{2.1}$$

where $\rho = \frac{1}{2}\alpha$ is the Weyl vector (α is the only root),

$$q = e^{\frac{i\pi}{k+2}} \tag{2.2}$$

and H_λ is the Cartan generator corresponding to λ. If one represents SU(2) as a 3-sphere of radius r embedded in $\mathbb{R}^4$, then the (classical) position of the brane is given by

$$x^4 = \frac{r}{2}\,\mathrm{tr}(t_{2\lambda}) = r\,\cos\frac{(2\lambda+1)\pi}{k+2}\,. \tag{2.3}$$

It is known that $r^2 = k$ for large k. Hence the world-volume of the (D2-) brane looks like a 2-sphere embedded in S^3 at the angle $\frac{(2\lambda+1)\pi}{k+2}$, see figure 1.

The fluctuations of these branes have been analyzed in the first paper of [3]. It turns out that they decompose as $1 \otimes \lambda \otimes \lambda = (\oplus_{j=0}^{2\lambda-1}\, j) \oplus (\oplus_{j=1}^{2\lambda}\, j) \oplus (\oplus_{j=1}^{2\lambda+1}\, j)$. The middle series is gauged away. The spectrum of the physical fluctuations is

$$m_j^2 = \frac{1}{k}\begin{cases} (j+1)(j+2) & \text{for } j = 0, 1, \dots 2\lambda - 1 \\ (j-1)j & \text{for } j = 1, 2, \dots 2\lambda + 1 \end{cases} \tag{2.4}$$

It contains a triplet of massless modes for $l = 1$, corresponding to the Goldstone modes of the broken rotational symmetry $\mathrm{SO}(4) \to \mathrm{SO}(3)$. All higher modes are massive, indicating the harmonic stability of the D2-brane.

[1] Here and in the following λ will denote the (half-integer) spin, hence the weight is 2λ.

Branes can be viewed also as a bound state of D0-branes. For small WZW branes ($\lambda \ll k$) the physics of the system is provided by the ordinary matrix model [4]. In [9, 10] another matrix model of WZW branes based on quantum symmetries was proposed. It was shown there that the model properly describes the static properties of *all* branes. For $G = \mathrm{SU}(2)$ the variables of the model consist of 4 matrices M^μ (for $\mu = 1, 0, -1, 4$, and $i, j, k = 1, 0, -1$), subject to the relations

$$\begin{aligned} F^l_L(M) &\equiv i(qM^4M^l - q^{-1}M^lM^4) - \epsilon^l_{ij}M^iM^j = 0 \\ F^l_R(M) &\equiv i(-q^{-1}M^4M^l + qM^lM^4) - \epsilon^l_{ij}M^iM^j = 0\,. \end{aligned} \tag{2.5}$$

Here ϵ^k_{ij} is the q-deformed epsilon-tensor, which is recalled in appendix A.1 along with the other q-deformed objects needed. These relations were obtained in [9] by requiring invariance under a "twisted" quantum symmetry $U_q(so(4))_{\mathcal{F}}$. The M^μ should be thought of as quantizations of the coordinate functions x^μ of the embedding space $\mathbb{R}^4$. They can also be written as 2×2 matrices in terms of the four (q-) Pauli matrices $\sigma_\mu = (1, -iq^{-1}\sigma_i)$

$$M = M^\mu \sigma_\mu \equiv M^4 - iq^{-1}M^D\,, \tag{2.6}$$

where $\mathrm{tr}_q(M^D) = 0$.[2][3] In this 2×2 notation, the relations (2.5) take the form $R_{21}M_2R_{12}M_1 = M_1R_{21}M_2R_{12}$ (in short-hand notation), which is known as reflection equation (RE). Here R_{12} is the so-called R-matrix of $U_q(su(2))$ in the fundamental representation. There is yet another way of writing these relations, which will be used extensively throughout this paper. We also introduce $\tilde{\sigma}_\mu = (1, iq\,\sigma_i)$ and

$$\tilde{M} = M^\mu \tilde{\sigma}_\mu \equiv M^4 + iqM^D\,. \tag{2.7}$$

We can then split $M\tilde{M}$ and $\tilde{M}M$ into trace and traceless parts $F_{L,R} = F^l_{L,R}\,\sigma_l$, $l = 1, 2, 3$,

$$M\tilde{M} = \mathrm{det}_q(M) + F_L(M)\,, \qquad \tilde{M}M = \mathrm{det}_q(M) + F_R(M)\,. \tag{2.8}$$

Notice that for $\mathfrak{g} = su(2)$, $\mathrm{det}_q(M) = \mathrm{tr}_q(M\tilde{M})/[2] = \mathrm{tr}_q(\tilde{M}M)/[2] = (M^4)^2 + M^iM^jg_{ij}$. Then (2.5) are equivalent to

$$F_L(M) = F_R(M) = 0\,. \tag{2.9}$$

The algebra defined by the relations (2.5) resp. (2.9) has two central elements, which provide the basis of the interpretation in terms of D-branes. The first one is the quantum determinant which is invariant under the full $U_q(so(4))_{\mathcal{F}}$ symmetry algebra. We shall use this to impose the constraint

$$\mathrm{det}_q(M) = r^2 \tag{2.10}$$

which in a sense defines a 3-sphere of radius r. For simplicity we shall take $r = 1$ from now on. The second central element is M^4, which is invariant only under the "vector" subalgebra $U_q(su(2))^V$ of $U_q(so(4))_{\mathcal{F}}$.

[2]Note that M^D coincides with the Dirac operators on the quantum sphere constructed by Bibikov and Kulish [12].

[3]Lower-case tr_q means q-trace for the spin 1/2 representation, see (A.4).

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The irreps of the RE algebra coincide with those of $U_q(su(2))$, and are labeled by a spin $\lambda \in \{0, 1/2, \ldots, k/2\}$. For a given irrep λ, each M^i is a $(2\lambda+1) \times (2\lambda+1)$ matrix, and the Casimir M^4 takes the value

$$M^4_\lambda = \frac{1}{[2]_q}\left(q^{2\lambda+1} + q^{-2\lambda-1}\right) = \frac{\cos\frac{(2\lambda+1)\pi}{k+2}}{\cos\frac{\pi}{k+2}} \,. \tag{2.11}$$

This fits nicely with (2.3), and allows to identify this irrep with the stable brane $D_{2\lambda}$. Further support for this identification and more details can be found in [10].

3. Degrees of freedom and symmetries

We imagine that in order to describe excitations of the branes, we need some kind of field theory living on the NC algebras defined above. As we do not have any general procedure applicable in this case, we shall naively follow certain guidelines stemming from the standard $D0$-brane matrix model [4] and other noncommutative gauge theories. These models show that it is useful to combine the matrix fields A^μ and the matrix background M^μ into a single variable which we shall call B^μ. These will be general matrices subject only to some reality conditions (to be defined later). In other word, we shall split

$$B^\mu = M^\mu + A^\mu \,, \tag{3.1}$$

where M^μ is given as in the previous section, and A^μ is arbitrary (but in a certain sense small if corresponding to excitations). The four matrices B^μ can again be assembled into a 2×2 matrix B, $\tilde{B}$ and we can define B^D, $F_{L,R}(B)$ according to (2.6)–(2.8). They shall transform as $B \to \pi(u^L) B \pi(Su^R)$ under $U_q(so(4))_{\mathcal{F}}$, where π is the spin 1/2 representation of $U_q(su(2))$.[4] In this paper we will impose the constraint

$$\det_q(B) = 1 \,, \tag{3.2}$$

setting the radius $r = 1$ for simplicity. This important point will be discussed in section 4.

The simplest nontrivial action is now [9]

$$S[B] = a_L S_L[B] + a_R S_R[B] = a_L \mathrm{Tr}_q(F_L(B)F_L(B)) + a_R \mathrm{Tr}_q(F_R(B)F_R(B)) \,. \tag{3.3}$$

for some constants $a_{L,R}$. These actions are by construction invariant under $U_q(so(4))_{\mathcal{F}}$. It is understood here that each B^μ is a matrix acting on some Hilbert space $\mathcal{H}$, which can be a spin λ representation of $U_q(su(2))$ for a fixed brane $D_{2\lambda}$, or a more general Hilbert space as discussed in section 3.1. Then Tr_q denotes the quantum trace (see appendix A.1) over both $\mathcal{H}$ and over the explicit 2×2 matrices; the internal trace can be interpreted as integral. The background configurations $B = M$ respect $F_L(M) = F_R(M) = 0$, thus they are solutions with $S[M] = 0$.

[4] One can show that $\tilde{B}$ transforms as B^{-1}, and

$$B\tilde{B} \to \pi(u_1^L) B\tilde{B} \pi(S(u_2^L)) \,, \qquad \tilde{B}B \to \pi(u_1^R) \tilde{B}B \pi(S(u_2^R)) \,,$$

which is the basis for writing down invariants.

Several remarks are in order. First, the quantum trace over $\mathcal{H}$ guarantees invariance under the following action of $u \in U_q(su(2))$

$$B^i_j \;\to\; Su_1\, B^i_j\, u_2\,, \tag{3.4}$$

where $u_{1,2} \in U_q(su(2))$ acts on $\mathcal{H}$ in the appropriate representations. For the solutions $B = M$ this transformation is equivalent [10] to the "vector" rotations $B \to \pi(u_1)B\pi(Su_2)$ for $u \in U_q(su(2))^V \subset U_q(so(4))_{\mathcal{F}}$, hence the action must be invariant under (3.4). Taking the classical trace over $\mathcal{H}$ instead of the quantum trace would violate this invariance.

Second, even though the action is constructed in terms of invariants of the quantum group $U_q(so(4))_{\mathcal{F}}$, the precise meaning of a quantum group symmetry in field theory is far from trivial. The point is that $S[B]$ is an *invariant expression* in the sense that it is invariant if $u \in U_q(so(4))_{\mathcal{F}}$ acts on each term using the coproduct, $u \triangleright S[B] = S[B]\;\epsilon(u)$ (here $\epsilon(u)$ is the counit). However the coproduct is nontrivial, hence it is *not* an invariant functional of the matrices: $S[u \triangleright B] \neq S[B]\;\epsilon(u)$. This will imply e.g. that we will not see certain zero modes later. It probably means that one should treat the B^μ's as some kind of "dressed" quantum fields rather than classical matrices.[5] One way of dealing with this problem has been proposed in [14] in a simpler context. We will ignore this issue in the present paper, and treat $S[B]$ in a naive way. This should be seen as a first step towards a fully consistent treatment.

Star structure. In order to have a meaningful theory of branes on SU(2), we must specify the appropriate reality constraints for the gauge fields B. This should be done in a way which is consistent with the solutions $B = M$, and compatible with the quantum group symmetries. Hence we cannot simply impose $B^\dagger = B^{-1}$ or $B^\dagger = B$, since M does not satisfy these constraints. We define here a different conjugation (star structure) respecting $\star^2 = 1$, $\star(XY) = \star Y \star X$, by

$$\star(X) := \pi(\hat{w})\hat{w}X^\dagger\hat{w}^{-1}\pi\left(\hat{w}^{-1}\right). \tag{3.5}$$

Here $X^\dagger = (X^*)^T$, i.e. $(X^i_j)^\dagger = (X^j_i)^*$, and x^* is the usual adjoint of an operator x acting on a Hilbert space. The element $\hat{w} \in U_q(su(2))$ is the "universal Weyl element" [15] of $U_q(su(2))$, which acts on representations like a (q-) mirror reflection of the weights.

It turns out (see appendix) that $\star M = \tilde{M}$, which[6] implies $\star(M_D) = M_D$, $\star(M^4) = M^4$. In component form, this star becomes $\star(M^i) = -M^i$ where now $\star(M^i) = \hat{w}(M^i)^*\hat{w}^{-1}$. We therefore impose the same condition globally on the dynamical degrees of freedom:

$$\star(B) = \tilde{B}\,, \tag{3.6}$$

which implies $\star(B_D) = B_D$, $\star(B^4) = B^4$. It is shown in the appendix that this reality condition is preserved under "vector rotations" $B \to \pi(u_1)B\pi(Su_2)$ for $u \in U_q(su(2))^V$, as well as under (3.4). One can now check that the determinant $\det_q(B) \propto \mathrm{tr}_q(B\tilde{B})$ is real,

[5] Using a covariant differential calculus rather than "component fields" B^μ would not change this issue: the B^μ can be understood as components of a one-form w.r.t. a frame, cp. [13]

[6] Also $\star(M) = M^{-1}$.

and so are the actions $\mathrm{Tr}_q(FF) = \mathrm{Tr}_q(F \star F)$ using (A.17) and

$$\star(F) = F\,. \tag{3.7}$$

More details are given in the appendix A.

3.1 Static multi-brane configurations

We also want to consider configurations of several branes and their excitations. Recall that a single static brane corresponds to an irreducible representation of RE, hence also of $U_q(su(2))$. Reducible representation $M = \oplus_\lambda M_\lambda$ therefore describe several branes, i.e.

$$M = \begin{pmatrix} M_{\lambda_1} & 0 & \dots & 0 \\ 0 & M_{\lambda_2} & \dots & 0 \\ & \dots & & \\ 0 & \dots. & 0 & M_{\lambda_N} \end{pmatrix}. \tag{3.8}$$

It is then natural to decompose the field B which describes the interaction and excitations of such a system in the form

$$B = \begin{pmatrix} B_{\lambda_1} & B_{\lambda_1\lambda_2} & \dots & B_{\lambda_1\lambda_N} \\ B_{\lambda_2\lambda_1} & B_{\lambda_2} & \dots & B_{\lambda_2\lambda_N} \\ & \dots & & \\ B_{\lambda_N\lambda_1} & \dots & B_{\lambda_N\lambda_{N-1}} & B_{\lambda_N} \end{pmatrix}, \tag{3.9}$$

reflecting that there are N distinct "positions" for the branes. Notice that $B_{\lambda\gamma} \in (\lambda \otimes \frac{1}{2}) \otimes (\gamma \otimes \frac{1}{2})^*$ while $B_\lambda \in (\lambda \otimes \frac{1}{2}) \otimes (\lambda \otimes \frac{1}{2})^*$. Requiring that single-brane configurations are consistent, we should take the reducible representation of $\hat{w}$, so that

$$\hat{w} = \begin{pmatrix} \hat{w}_{\lambda_1} & 0 & 0 \\ & \dots & \\ 0 & 0 & \hat{w}_{\lambda_N} \end{pmatrix}. \tag{3.10}$$

Also the quantum trace should then be defined block-wise[7]. We therefore extend the star operation on B as

$$\star(B^4_{\lambda\gamma}) = B^4_{\gamma\lambda}\,, \qquad \star(B^D_{\lambda\gamma}) = B^D_{\gamma\lambda} \tag{3.11}$$

hence $B^i_{\lambda\gamma} = -\star(B^i)_{\gamma\lambda} = -\hat{w}_\gamma\,(B^i_{\gamma\lambda})^*\,\hat{w}_\lambda^{-1}$ where $*$ means the usual adjoint of an operator.

(This is not the only possibility: one could also take a representation of a given size N, which might correspond to the number of $D0$-branes.).

4. Brane dynamics

We now turn to the dynamics of these branes. Hence we consider fluctuations of the form

$$B = (M^4 + A^4) - iq^{-1}(M^D + A^D)\,, \qquad \tilde{B} = (M^4 + A^4) + iq(M^D + A^D)\,. \tag{4.1}$$

[7] In principle, there is an equivalence map $D_{2\lambda} \to D_{2(k/2-\lambda)}$, defining some automorphism α. One could then also impose $\star(B^i_D) = -\alpha(B^i_D)$ etc.

For the off-diagonal entries in (3.9) the background is $M_{\lambda_1 \lambda_2} = 0$. All the fluctuations around the background are in F_L, due to the determinant constraint. Recalling that $F_L(M) = 0$, they are

$$F_L(B) = i(qA^4 M^D - q^{-1} M^D A^4) + Q \triangleright A^D - (Q \triangleright A^D)^{(4)} + O(A^2)\,, \tag{4.2}$$

where

$$(Q \triangleright A^D)^{(4)} \equiv \mathrm{tr}_q(Q \triangleright A^D)/[2] \tag{4.3}$$

$$Q \triangleright A^D \equiv i(qM^4 A^D - q^{-1} A^D M^4) + M^D A^D + A^D M^D\,. \tag{4.4}$$

We have split F_L into several pieces in (4.2) according to their importance for the calculations we are going to perform. We shall elaborate on this in the course of the paper. Here we just note that the first term in (4.2) can be neglected, and the most important part will be $Q \triangleright A^D - (Q \triangleright A^D)^{(4)}$. We can get similar formulae for F_R.

We decided to set $\det_q(B) = r^2 = 1$ in this paper. There are several reasons for this. First of all the constraint reduces the number of the excitations by one. The radius r is in principle determined by the background, and can easily be reinserted if desired. To first order in A's, the constraint $\det_q(B) = 1$ implies

$$M^4 A^4 + A^4 M^4 + (Q \triangleright A^D)^{(4)} = 0\,, \tag{4.5}$$

which determines A^4 in terms of A^D. Furthermore, other calculations [3] show that there are massless excitation modes corresponding to non-trivial rotations of the spherical D2-brane inside $S^3 = \mathrm{SU}(2)$. Such a rotation (even an infinitesimal one, since the radius of SU(2) is finite here) changes the x^4 resp. B^4 coordinate, thus we need A^4 to be determined from $\det_q(B) = 1$. In this paper we shall use only the linear approximation (4.5), leaving a complete implementation of the constraint $\det_q(B) = 1$ open. Notice also that (4.5) is invariant under gauge transformation while $\det_q(B)$ is not.

Let us summarize our setup: we assume a background respecting $F_L(M) = F_R(M) = 0$, and impose $\det_q(B) = 1$. From (3.3) and the expansion (4.1) the action for excitations becomes $a_L \mathrm{Tr}_q F_L(B)^2 + a_R \mathrm{Tr}_q F_R(B)^2$ with $F_L(B)$ given by (4.2), and an analogous expression for F_R. Since both terms will give basically the same results, we shall work mostly with $\mathrm{Tr}_q(F_L(B)^2)$ and set $a_L = 1$. Thus we will study the action

$$S_L = \mathrm{Tr}_q(F_L(B)^2)\,. \tag{4.6}$$

4.1 Spectrum of single-brane fluctuations

In this subsection we shall determine the excitation spectrum on a single brane. We will consider the action as an ordinary functional of the fluctuation matrices A^μ, and use a harmonic expansion for the matrix entries. While the intermediate steps and formulae in the calculation are adapted to this new context, we shall see that the expansion has very similar properties to ordinary harmonic analysis.

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The result of the calculations will be the masses of brane excitations. For low harmonics, they will differ from the known results [3]. The most important point is that there there will not be any massless modes corresponding to rigid rotations of the brane. We shall comment on this point in the end of this subsection. Apart from the zero modes corresponding to gauge invariance all masses are positive, hence the branes are stable in our model.

In order to facilitate the calculations we shall work here only in the large k expansion, which is good enough to exhibit the main properties of the spectrum. In this limit, all q-tensors appearing in (2.5),(3.3) can be replaced by their undeformed counterparts; thus $g_{ij} \to \delta_{ij}$ and $\varepsilon_{ij}^{l} \to \varepsilon_{ijl}^{c}$ where ε_{ijl}^{c} is the ordinary antisymmetric tensor. Then $F_L = F_R = 0$ are solved by $M^l = -i(q-q^{-1})M^4 x^l$ where $[x^i, x^j] = i\varepsilon_{ijl}^{c} x^l$, i.e. the x's satisfy the ordinary $su(2)$ algebra in the spin λ representation (corresponding to the brane). Assuming that the spin j of the fields respects $j \ll k$ we can compare the contributions of the various terms, counting powers of k. Then we get $Q \triangleright A^D \sim A^D\ M^4/k$ and $A^4 \sim A^D/k$, thus

$$i\left(qA^4 M^D - q^{-1} M^D A^4\right) \sim \frac{M^4}{k^3} A^D \tag{4.7}$$

and we can neglect this term compared to $Q \triangleright A^D$. Moreover $S_L = S_R$ in the case of a single brane $D_{2\lambda}$, since M^4 is central. Thus we shall discuss only S_L.

Therefore the mass matrix for excitations we are going to consider for a brane $D_{2\lambda}$ is

$$S_L = \mathrm{Tr}_q(F_L F_L) = \mathrm{Tr}_q\left[\left(Q \triangleright A^D\right)\left(Q \triangleright A^D\right) - \left(Q \triangleright A^D\right)^{(4)}\left(Q \triangleright A^D\right)^{(4)}\right]. \tag{4.8}$$

The star was omitted here, since $\star(Q \triangleright A^D) = Q \triangleright A^D$ due to the reality constraints. Using the fact that the quantum trace is cyclic[8] with respect to M^D, this can be rewritten as

$$S_L = \mathrm{Tr}_q\left[A^D\left(Q^2 \triangleright A^D\right) - \left(Q \triangleright A^D\right)^{(4)}\left(Q \triangleright A^D\right)^{(4)}\right], \tag{4.9}$$

where

$$Q^2 \triangleright A^D = \left(\alpha^2 + 2\beta\right) A^D - \alpha\left(M^D A^D + A^D M^D\right) + 2M^D A^D M^D \tag{4.10}$$

using the characteristic equation (A.23) on the brane,

$$\begin{aligned}
(M^D)^2 &= \alpha M^D + \beta\,, \\
\alpha &= -hM^4\,, \qquad \beta = 1 - (M^4)^2\,, \\
h &= i(q - q^{-1})\,.
\end{aligned} \tag{4.11}$$

The action for the quadratic fluctuations has the following infinitesimal gauge invariance

$$\delta A^D = i\left(M^D f - f M^D\right) \tag{4.12}$$

for any $f \in \mathrm{Mat}(2\lambda+1)$, i.e. $(Q \triangleright \delta A^D) = 0$. This gauge invariance is not a standard consequence of symmetries of the action but rather its equations of motion $F_L(M) = 0$ and the fact that the action is a functional of F_L. Notice that after (4.12) we get (for

[8]This follows from the explicit realization $M = (\pi_\gamma \otimes \pi_{\frac{1}{2}})(\mathcal{R}_{21}\mathcal{R}_{12})$ where $\mathcal{R}$ is the universal R matrix of $U_q(su(2))$, which commutes with $q^H \otimes q^H$

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infinitesimal f) $B = M + A \to e^{-if} M e^{if} + A$ thus terms linear in A in the expansion (4.2) of $F_L(B)$ are not changed because $e^{if} M e^{-if}$ also solves equations of motion. The argument works for any functional depending solely on F_L or F_R.

4.1.1 Harmonic expansion

It is very important to choose a convenient basis for the gauge fields A. The latter are elements of $\mathrm{Mat}(2\lambda+1) \otimes \mathrm{Mat}(2)$ and can be represented in the following manner: $A^D = A_{(1)} + A_{(2)} + A_{(g)}$ where

$$\begin{aligned} A_{(1)} &= M^D f + f M^D \,, \\ A_{(2)} &= M^D f' M^D - (M^D f' M^D)^{(4)} \\ A_{(g)} &= i(M^D f'' - f'' M^D)\,, \end{aligned} \tag{4.13}$$

where $f,\ f'$ are “scalar fields” with values in $\mathrm{Mat}(2\lambda+1)$. A^4 is then determined by (4.5). We can discard the gauge degrees of freedom $A_{(g)}$ as discussed above. Moreover the fields $A_{(i)}$ satisfy the reality constraints (3.6) provided $\star(f) = \hat{w} f^* \hat{w}^{-1} = f$.

We now expand $f \in \mathrm{Mat}(2\lambda+1)$ in terms of harmonics, i.e. eigenvalues of a suitable Laplacian. The following definition turns out to be useful:

$$\Delta f = [\lambda_i, f]\lambda^i = -\frac{1}{2}[\lambda_i, [\lambda^i, f]]\,, \tag{4.14}$$

where $M^D = \lambda_i \sigma^i$, noting that $\lambda_i \lambda^i$ is central. From the algebra relations (4.11), it follows that $J_i = \frac{\lambda_i}{\alpha}$ satisfies $i\varepsilon JJ = J$. Furthermore, $\star(\Delta f) = \Delta \star f$. In order to simplify the calculations below, we shall work in the leading order in $1/k$ for the rest of this subsection. Then the eigenvalues of Δ are to lowest order given by

$$\Delta f_{jm} = -\frac{1}{2}\alpha^2\, j(j+1)\, f_{jm} \qquad \left(+o\left(\frac{1}{k}\right)\right). \tag{4.15}$$

Moreover we can normalize the harmonics in the standard way[9] $\mathrm{Tr}_q(f_{jm} f_{j'm'}) = \delta_{jj'}\delta_{mm'}$. Any $f \in \mathrm{Mat}(2\lambda+1)$ can now be represented as

$$f = \sum_{jm} a_{jm} f_{jm}\,. \tag{4.16}$$

With this we get

$$A_{(2)} = M^D f' M^D - (\beta + \Delta) f'\,, \tag{4.17}$$

where Δ becomes a number if f' is a harmonic.

Notice that contribution to (4.9) of different harmonics are orthogonal to each other. This can be seen as follows: after expanding A^D in harmonics $(f,\ f')$, any term in (4.9) is of the form (using cyclicity for M^D)

$$\mathrm{Tr}_q\left(\left(M^D\right)^m f \left(M^D\right)^n f'\right). \tag{4.18}$$

[9] Because the laplacian is self-adjoint w.r.t. the inner product $\mathrm{Tr}_q(f \star g)$.

Using the characteristic equation (4.11), one can furthermore simplify these to

$$\mathrm{Tr}_q\left(ff'\right), \qquad \mathrm{Tr}_q\left(fM^D f'\right), \qquad \mathrm{Tr}_q\left(M^D ff'\right), \qquad \mathrm{Tr}_q(M^D fM^D f'). \tag{4.19}$$

The second and the third one vanish due to Tr_q, while the last one is proportional to $\mathrm{Tr}_q(ff')$ since $\mathrm{Tr}_q(M^D fM^D f') = \mathrm{Tr}_q((M^D fM^D)^{(4)} f') = (\beta+\Delta)\mathrm{Tr}_q(ff')$. This vanishes if f and f' are different harmonics.

To summarize, we can restrict ourselves to quadratic fluctuations of the form

$$A^D = a_1 A_{(1)} + a_2 A_{(2)}, \tag{4.20}$$

where $a_i \in \mathbb{R}$ and $f = f'$, $\Delta f = -\frac{1}{2}\alpha^2\, j(j+1)\, f$. Then the action (4.8) becomes

$$S = a_i a_j\ \mathrm{Tr}_q\left(Q \triangleright A^D_{(i)}\, Q \triangleright A^D_{(j)} - \left(Q \triangleright A^D_{(i)}\right)^{(4)}\left(Q \triangleright A^D_{(j)}\right)^{(4)}\right). \tag{4.21}$$

The calculation now proceeds using the same tricks as above. It is convenient to introduce the matrix of normalizations[10] for the modes $A^D_{(i)}$,

$$G_{ij} \equiv \mathrm{Tr}_q(A^D_{(i)} A^D_{(j)}) = \begin{pmatrix} 4\beta + 2\Delta, & 2\alpha(\beta+\Delta) \\ 2\alpha(\beta+\Delta), & \alpha^2\beta + (\alpha^2 - 2\beta)\Delta - \Delta^2 \end{pmatrix}. \tag{4.22}$$

Here Δ will always stand for its eigenvalue on f resp. f'. We furthermore need

$$\begin{aligned}
\left(Q \triangleright A_{(1)}\right)^{(4)} &= (4\beta + 2\Delta) f, \\
\left(Q \triangleright A_{(2)}\right)^{(4)} &= 2\alpha(\beta+\Delta) f', \\
\left(Q^2 \triangleright A_{(1)}\right)^{D} &= 4\beta A_{(1)} + 2\alpha A_{(2)} \\
\left(Q^2 \triangleright A_{(2)}\right)^{D} &= (\alpha^2 - 2\Delta) A_{(2)} + (2\alpha\beta + \alpha\Delta) A_{(1)}.
\end{aligned} \tag{4.23}$$

Introducing

$$\tilde{Q} = \begin{pmatrix} 4\beta, & 2\alpha\beta + \alpha\Delta \\ 2\alpha, & \alpha^2 - 2\Delta \end{pmatrix} \tag{4.24}$$

the action (4.21) becomes

$$a_i a_j \left[G\tilde{Q} - \begin{pmatrix} 4\beta + 2\Delta \\ 2\alpha(\beta+\Delta) \end{pmatrix} (\, 4\beta + 2\Delta, \quad 2\alpha(\beta+\Delta)\,)\right]_{ij} = a^T\, GT\, a \tag{4.25}$$

where

$$T = \begin{pmatrix} -2\Delta, & -\alpha\Delta \\ 2\alpha, & \alpha^2 - 2\Delta \end{pmatrix}. \tag{4.26}$$

The matrix T has eigenvalues

$$\begin{aligned}
t_1 &= \alpha^2\, l^2, & l &= 1, 2, \ldots, 2\lambda \\
t_2 &= \alpha^2\, (l+1)^2, & l &= 0, 1, \ldots, 2\lambda
\end{aligned} \tag{4.27}$$

[10]Notice that G_{ij} is not singular except one case $j = 0$. In this case $A_{(1)}$ and $A_{(2)}$ are dependent thus we must remove one them.

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which are the mass spectrum of gauge fields.[11] Each value has the usual $2l+1$ degeneracy, and $l=0$ must be excluded from the first series because $A_{(1)}$ and $A_{(2)}$ coincide in that case. In particular, all masses are positive, reflecting the stability of the branes.

This spectrum should be compared with (2.4). We see that the result is very close (including the correct scaling dependence on k), but for small l it is not the same. Most notably we are missing the triplet of zero-modes in (2.4), which correspond to the nontrivial rotation of the 2-branes in S^3. The lack of rotational zero-modes indicates that our action is *not* invariant under SO(4). This may seem strange, because the action was constructed to be invariant under a quantum version of that symmetry, $U_q(so(4))_{\mathcal{F}}$. This was successful in the sense that it leads quite directly to equations of motion (RE) which have the correct brane solutions. We recall however the discussion in section 3 that this symmetry is formal — in our naive treatment as ordinary matrix model — and not a symmetry of the action functional: $S[u\triangleright B] \neq S[B]\,\varepsilon(u)$. This indicates that a fully consistent way of implementing this quantum symmetry is still to be found.

4.2 Strings stretched between branes

We now discuss several branes, and calculate masses of states which mediate interactions between them. We must therefore consider reducible representations of M, and include the "off-diagonal" sectors $B_{\gamma\lambda}$ which connect different branes.

Assume that there are 2 branes $D_{2\gamma}$ and $D_{2\lambda}$ present. We want to calculate the lowest energy (mass) for a "string" connecting these 2 branes. This scenario is described by the matrix

$$B = \begin{pmatrix} M_\gamma & A_{\gamma\lambda} \\ A_{\lambda\gamma} & M_\lambda \end{pmatrix}. \tag{4.28}$$

One could now in principle do a similar analysis as for the single-brane modes. To simplify the calculation, we shall compute here the ground state energy (mass) only, using a somewhat different approach.

To proceed, it is useful to introduce a basis of eigenvectors of the "Dirac operators" M^D. We consider here M^D as acting on $(\gamma\otimes\frac{1}{2})$ from the left and on $(\lambda\otimes\frac{1}{2})^*$ from the right. The star refers here to the way $U_q(su(2))$ acts, and will be explained below. The left eigenvalues of M^D are [12]

$$M^D\,|\gamma\alpha,m\rangle = c_\gamma^\alpha\,|\gamma\alpha,m\rangle \quad \in \quad \left(\gamma\otimes\frac{1}{2}\right), \tag{4.29}$$

where $\alpha=\pm\frac{1}{2}$, $m\in\{-\gamma,\dots,\gamma\}$, $c_\gamma^{\pm 1/2} = \pm h\,[\gamma+(1\mp 1)\frac{1}{2}]_{q^2}$ and $h=i(q-q^{-1})$. This follows e.g. from the characteristic equation (4.11) of M^D, which implies that there are also right eigenvectors with the same eigenvalues:

$$\langle\lambda\beta,n|\;M^D = c_\lambda^\beta\,\langle\lambda\beta,n|\,. \tag{4.30}$$

[11]To see this, assume that we use an orthonormal basis $A^o_{(i)}$ instead of (4.13) and the expansion (4.20) reads $A^D = b_1 A^o_{(1)} + b_2 A^o_{(2)}$. Then we can write $G = g^T g$ and $b_i = g_{ij} a_j$. Thus (4.25) is $a^T\;GT\;a = b^T\;g\;T\;g^{-1} b$, and the eigenvalues of $g\;T\;g^{-1}$ and T are the same yielding the masses.

To understand the transformation properties of these eigenvectors, we recall the basic fact that M^D commutes with the coproduct $\Delta(u) = u_1 \otimes u_2$ of $u \in U_q(su(2))$:

$$(u_1 \otimes u_2)M^D = M^D(u_1 \otimes u_2)\,, \tag{4.31}$$
$$M^D(Su_2 \otimes Su_1) = (Su_2 \otimes Su_1)M^D\,, \tag{4.32}$$

where the appropriate representations $(\pi_\gamma, \pi_{\frac{1}{2}})$ are understood.[12] This implies that the left eigenvectors $|\gamma\alpha, m\rangle \in (\gamma \otimes \frac{1}{2})$ transform under the following action of $U_q(su(2))$ (below $\alpha, \beta = \pm\frac{1}{2}$):

$$u \triangleright (|p\rangle \otimes |\alpha\rangle) := u_1|p\rangle \otimes u_2|\alpha\rangle \quad \in \quad \left(\gamma \otimes \frac{1}{2}\right). \tag{4.33}$$

This justifies the notation in (4.29) using the quantum number m. Similarly, (4.32) implies that the right eigenvectors $\langle\lambda\beta, n| \in (\lambda \otimes \frac{1}{2})^*$ transform under the following "dual" action of $U_q(su(2))$:

$$u \triangleright (\langle l| \otimes \langle\beta|) := \langle l|Su_2 \otimes \langle\beta|Su_1 \quad \in \quad \left(\lambda \otimes \frac{1}{2}\right)^*. \tag{4.34}$$

Therefore the matrices

$$\xi_{\gamma\alpha m;\lambda\beta n} := |\gamma\alpha, m\rangle\langle\lambda\beta, n| \tag{4.35}$$

transform as

$$u \triangleright \xi_{\gamma\alpha m;\lambda\beta n} = (u_1 \otimes u_2)\xi_{\gamma\alpha m;\lambda\beta n}(Su_4 \otimes Su_3) \quad \in \quad \left(\gamma \otimes \frac{1}{2}\right) \otimes \left(\lambda \otimes \frac{1}{2}\right)^*, \tag{4.36}$$

and one can decompose them accordingly into irreps of $U_q(su(2))$. Furthermore, $\star(M^D) = M^D$ implies that $\star\xi_{\gamma\alpha m;\lambda\beta n}$ is also "eigenmatrix" of M^D with flipped left and right eigenvalues, and one can check (see appendix A) that

$$\star(u \triangleright \xi_{\gamma\alpha m;\lambda\beta n}) = u \triangleright \star\xi_{\gamma\alpha m;\lambda\beta n} \tag{4.37}$$

provided u is in the following "real" sector of the rotation algebra (see also (A.25))

$$\mathcal{G}^V := \left\{u = \theta u^* = \hat{w}(Su^*)\hat{w}^{-1}\right\}. \tag{4.38}$$

This means that the star $\star$ restricts consistently to irreps of $U_q(su(2))$.

Armed with these tools, we return to the gauge fields $A_{\gamma\lambda}$ which a priori are arbitrary matrix valued fields in $(\gamma \otimes \frac{1}{2}) \otimes (\lambda \otimes \frac{1}{2})^*$. We can therefore expand them in the basis of eigenvectors of the Dirac operator (4.29):

$$A_{\gamma\lambda} = \sum a(\gamma\alpha, m; \lambda\beta, n)\ \xi_{\gamma\alpha m;\lambda\beta n} \tag{4.39}$$

for arbitrary $a(\gamma\alpha, m; \lambda\beta, n)$. Note that in general, the matrices $\xi_{\gamma\alpha m;\lambda\beta n}$ have non-vanishing trace, i.e. in general $A^4 \neq 0$ in the above expansion. However, we are interested only in the ground states (which we assume to be the minimal spin states) here. Assume furthermore that $\gamma \geq \lambda + 1$. Then the spin of the ground state is $(\gamma - \lambda - 1)$. This implies

[12] This follows from the explicit realization $M = (\pi_\gamma \otimes \pi_{\frac{1}{2}})(\mathcal{R}_{21}\mathcal{R}_{12})$

that there is a unique such multiplet, and it must be in $1 \otimes \gamma \otimes \lambda \subset (\gamma \otimes \frac{1}{2}) \otimes (\lambda \otimes \frac{1}{2})^* = ((\gamma - \frac{1}{2}) \otimes (\lambda + \frac{1}{2})) \oplus \ldots = (\gamma - \lambda - 1) \oplus \ldots$ where dots denote the higher spin states. Therefore the ground state has the form

$$A^D_{\gamma\lambda} = A_{\gamma\lambda} = \sum a\left(\gamma - \frac{1}{2}, m; \lambda + \frac{1}{2}, n\right) \left|\gamma - \frac{1}{2}, m\right\rangle \left\langle \lambda + \frac{1}{2}, n\right| \tag{4.40}$$

with $A^4 = 0$, since the singlet in the spinor part does not enter.[13]

We can now calculate the eigenstates of the mass matrix for the ground states of the off-diagonal excitations in (4.28). We shall concentrate on S_L, and one can check that S_R gives the same result.

To get a real gauge field one must also include the conjugate term $A_{\lambda\gamma}$. Hence we should consider

$$\mathrm{Tr}_q \left[(Q \triangleright A^D)_{\gamma\lambda} * (Q \triangleright A^D)_{\gamma\lambda}\right] + (\lambda \leftrightarrow \gamma) \tag{4.41}$$

for fields of the form (4.40). Since for the ground states ξ is traceless as discussed above, we can ignore the terms $(Q \triangleright A^D)^{(4)}$ in the action (4.8). It is easy to see that

$$Q \triangleright \xi_{\gamma\alpha m;\lambda\beta n} = \left[i(qM^4_\gamma - q^{-1}M^4_\lambda) + h(c^\alpha_\gamma + c^\beta_\lambda)\right] \xi_{\gamma\alpha m;\lambda\beta n}\,, \tag{4.42}$$

where $\alpha, \beta = \pm 1/2$ according to signs appearing in (4.29). Q is not hermitean so the eigenvalue is not real. We also have

$$\star(Q \triangleright \ \xi_{\gamma\alpha m;\lambda\beta n}) = [i(qM^4_\gamma - q^{-1}M^4_\lambda) + h(c^\alpha_\gamma + c^\beta_\lambda)]^* \ \star \xi_{\gamma\alpha m;\lambda\beta n}\,. \tag{4.43}$$

Thus the mass matrix for these excitations $A^D = \xi$ has the following form

$$Q \triangleright A^D \star (Q \triangleright A^D) = m^2 \ \xi \star \xi \tag{4.44}$$

with eigenvalues

$$m^2 = |i(qM^4_\gamma - q^{-1}M^4_\lambda) + h(c^\alpha_\gamma + c^\beta_\lambda)|^2\,. \tag{4.45}$$

This is indeed large for $\alpha = \beta$ and small for $\alpha = -\beta$, as expected. Hence for $A^D = \xi = |\gamma - \frac{1}{2}, m\rangle\langle\lambda + \frac{1}{2}, n|^*$, the mass squared is

$$m^2 = |i(qM^4_\gamma - q^{-1}M^4_\lambda) + h([\lambda]_{q^2} - [\gamma + 1]_{q^2})|^2 = 4\sin^2\frac{(\gamma - \lambda)\pi}{k + 2} \tag{4.46}$$

using (2.11). For large k and $(\gamma - \lambda) \ll k$,[14] this is

$$m^2 \approx 4\left(\frac{(\gamma - \lambda)\pi}{k + 2}\right)^2\,, \tag{4.47}$$

which is indeed the (arc) length squared of the string stretched between branes, up to corrections of order $\frac{1}{k^2}$.

It is quite remarkable that this simple matrix model correctly reproduces the curvature effects of the underlying space S^3. This nicely supports the basic idea of our approach, which is to describe the dynamics of $D0$ branes in terms of a matrix model based on

[13] This argument requires the detailed discussion of the transformation properties given above.

[14] Recall that $\gamma \in [0, k/2]$.

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certain "quantum" symmetries. It would of course be very interesting to check whether the deviation of the ground state energy (4.46) from the arc-length is in agreement with string theory, perhaps due to the B field which is not closed.

The attentive reader might have noticed that the calculation presented here is not valid for $\gamma = \lambda + 1/2$. We expect that the result would be the same, which could e.g. be verified using an approach similar as in section 4.1.

5. Discussion

In this paper we analyzed the dynamics of the matrix model for D-branes on SU(2) which was proposed in [9, 10], based on noncommutative algebras related to quantum groups. This was motivated by the algebra of boundary operators which involves the $6j$-symbol of $U_q(su(2))$, and exhibits a particular truncation pattern related to that quantum group [5]. Our matrix model was designed to work for finite k as opposed to the ones found in [5].

The main result of this paper is to show that this model also gives a reasonable description for the dynamics of noncommutative D-branes. In particular, the branes turn out to be stable in our model. In doing so we tried to keep things simple, and stayed as close as possible to the usual treatment of matrix models. This includes ignoring some inconsistencies, in particular in the context of the quantum group symmetries. The price to pay is that we apparently do not get everything right, for example our model lacks the zero-modes associated to the global rotations. This could probably be cured if one would find a consistent implementation of the symmetries on a "quantum" level.

Let us summarize the main merits of the proposed models, including the results of [9, 10]:

- the geometric properties (quantized position, radius, ...) of the branes as embedded in the group manifold are reproduced, including the symmetry $\lambda \leftrightarrow k/2 - \lambda$
- each brane is a "fuzzy" (noncommutative) space with the correct fusion rules and truncations in the spectrum of harmonics
- the energy $\dim_q(V_\lambda)$ of a brane $D_{2\lambda}$ is naturally obtained by adding $\mathrm{Tr}_q(1)$ to the action
- the ground state energy for strings stretched between different branes seems (essentially) correct
- each brane is stable, with mass spectrum close to the correct one.

In view of all this, we believe that the models are very interesting as they are, being perhaps the first matrix models which describe branes on a curved background.

Of course it would be useful to extend our results to other directions. For $G = \mathrm{SU}(N)$, $N > 2$ the zoo of branes in the WZW model is much richer than for $G = \mathrm{SU}(2)$, for example there are so-called twisted branes [11]. One could also try to analyze the coset WZW models. There are plenty of results concerning ordinary matrix models for these systems [16] which can be used as guidelines for the construction of quantum matrix models.

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However, we should also emphasize that the model as treated here gives an incorrect mass spectrum of single branes, in particular the rotational zero modes are missing, which is a serious drawback. One should also keep in mind that there is some freedom in defining the action, in particular related to the constraint. For example, instead of imposing $\det_q(B) = const$ one could replace this by its linearized form (4.5), $M^4A^4 + A^4M^4 + (Q \triangleright A^D)^{(4)} = 0$, which is $U_q(so(4))$ invariant. Then $\mathrm{tr}_q(B\tilde{B}) = \mathrm{tr}_q(M\tilde{M}) + \mathrm{tr}_q(A\tilde{A})$ is another possible term in the action, which is quadratic in A. However, this would not change the main features of our results. Of course one could also consider higher-order terms of F in the action. Furthermore, the choice of reality conditions may affect the physical content of the models. While this is largely dictated by the reality property (3.6) of the solutions $B = M$, there may be other possibilities how to extend this to the fluctuations, perhaps requiring additional terms in the (real) action. However the main problem of the present approach seems to be the lack of a consistent implementation of this quantum group invariance beyond the formal level. This could be fixed by "quantizing" the matrices in an appropriate way (cp. [14]), or by some kind of symmetry-respecting Seiberg-Witten map. Finding a formalism which provides that is a challenge for the future, which seems worthwhile to pursue.

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A. Technical details

A.1 Basic properties of $U_q(su(2))$

The basic relations of the Hopf algebra $U_q(su(2))$ are

$$[H, X^\pm] = \pm 2X^\pm\,, \qquad [X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}} = [H]_q\,, \tag{A.1}$$

where the q-numbers are defined as $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The action of $U_q(su(2))$ on a tensor product of representations is encoded in the coproduct

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H \\ \Delta(X^\pm) &= X^\pm \otimes q^{-H/2} + q^{H/2} \otimes X^\pm\,. \end{aligned} \tag{A.2}$$

We use the Sweedler-notation $\Delta(u) = u_1 \otimes u_2$, where a summation convention is understood. The antipode and the counit are given by

$$\begin{aligned} S(H) &= -H\,, \qquad S(X^+) = -q^{-1}X^+\,, \qquad S(X^-) = -qX^-\,, \\ \epsilon(H) &= \epsilon(X^\pm) = 0\,. \end{aligned} \tag{A.3}$$

The quantum trace of an operator A acting on a representation of $U_q(su(2))$ is defined by

$$\mathrm{tr}_q(A) = \mathrm{tr}(A\pi(q^{-H}))\,, \tag{A.4}$$

where π denotes the representation. It has the important invariance property $\mathrm{tr}_q(\pi(u_1)$ $A\pi(Su_2)) = \mathrm{tr}_q(A)\epsilon(u)$ for any $u \in U_q(su(2))$. This is based on the identity

$$S^2(u) = q^{-H} u\, q^H \tag{A.5}$$

for $u \in U_q(su(2))$, which is easy to check.

Invariant tensors. The q-deformed sigma-matrices are given by

$$\sigma_{-1} = \begin{pmatrix} 0 & q^{1/2}\sqrt{[2]_q} \\ 0 & 0 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 0 \\ -q^{-1/2}\sqrt{[2]_q} & 0 \end{pmatrix}. \tag{A.6}$$

They satisfy

$$\sigma_i\sigma_j = -\epsilon_{ij}^k\,\sigma_k + g_{ij} \tag{A.7}$$
$$\pi(u_1)\sigma_i\pi(Su_2) = \sigma_j\pi_i^j(u)\,, \tag{A.8}$$

for $u \in U_q(su(2))$, where ϵ_{ij}^k is defined below, and π denotes the appropriate representation.

The invariant tensor g^{ij} for the spin 1 representation satisfies by definition

$$\pi_k^i(u_1)\,\pi_l^j(u_2)\,g^{kl} = \epsilon(u)g^{ij} \tag{A.9}$$

for $u \in U_q(su(2))$. It is given by

$$g^{1-1} = -q^{-1}\,, \qquad g^{00} = 1\,, \qquad g^{-11} = -q\,, \tag{A.10}$$

all other components are zero. Then $g_{ij} = g^{ij}$ satisfies $g^{ij}g_{jk} = \delta_k^i$, and $g^{ij}g_{ij} = q^2 + 1 + q^{-2} = [3]_q$.

The Clebsch-Gordon coefficients for $(3) \subset (3) \otimes (3)$, i.e. the q-deformed structure constants, are given by

$$\begin{aligned} \epsilon_1^{10} &= q^{-1}\,, & \epsilon_1^{01} &= -q\,, \\ \epsilon_0^{00} &= -(q - q^{-1})\,, & \epsilon_0^{1-1} &= 1 = -\epsilon_0^{-11}\,, \\ \epsilon_{-1}^{0-1} &= q^{-1}\,, & \epsilon_{-1}^{-10} &= -q\,, \end{aligned} \tag{A.11}$$

and $\epsilon_{ij}^k := \epsilon_k^{ij}$. They have been normalized such that $\sum_{ij}\epsilon_{ij}^n\epsilon_m^{ij} = [2]_{q^2}\delta_m^n$.

A.2 Star structure

We give some more details on the star structure here. Recall the universal Weyl element [15] w for $U_q(su(2))$, which satisfies

$$\begin{aligned} \Delta(w) &= \mathcal{R}^{-1} w \otimes w = w \otimes w\mathcal{R}_{21}^{-1}\,, \\ wuw^{-1} &= \theta S^{-1}(u)\,, \qquad w^2 = v\,, \end{aligned} \tag{A.12}$$

where v is a Casimir and θ the Cartan-Weyl involution. For $U_q(su(2))$ we have

$$w\ X_{\pm}w^{-1} = -q^{\pm 1}X_{\mp}, \quad w\ Hw^{-1} = -H\,. \tag{A.13}$$

Using this we define a rescaled $\hat{w} = wc$ where c is a suitable Casimir,[15] such that

$$\hat{w}^2 = 1\,, \qquad \hat{w}^{\dagger} = \hat{w}^{-1} = \hat{w}\,. \tag{A.14}$$

The $\star$ is then defined as in (3.5),

$$\star(X) := \pi(\hat{w})\hat{w}X^{\dagger}\hat{w}^{-1}\pi(\hat{w}^{-1})\,, \tag{A.15}$$

where $\pi(\hat{w})$ acts on the external indices of the matrix X and $\hat{w}$ on the "internal" space. It satisfies $\star\star = \mathrm{id}$ due to (A.14), and

$$\star(\mathrm{tr}_q(X)) = \mathrm{tr}_q(\star(X)) \tag{A.16}$$

since $q^H\hat{w}q^H = \hat{w}$. Then for any such matrices X, Y, one has

$$\star(\mathrm{tr}_q(XY)) = \mathrm{tr}_q(\star(XY)) = \mathrm{tr}_q(\star(Y)\star(X))\,. \tag{A.17}$$

Using $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}^{-1}$ together with (A.12), it follows that

$$\star(M) = M^{-1}\,. \tag{A.18}$$

Moreover, $\star(B) = \tilde{B}$ implies that

$$[2]\star(B^4) = \star\mathrm{tr}_q(B) = \mathrm{tr}_q(\star(B)) = \mathrm{tr}_q(\tilde{B}) = [2]B^4\,, \tag{A.19}$$

and together with $\star(B) = \star(B^4) + iq\star(B_D) = \tilde{B} = B^4 + iqB_D$ it follows that

$$\star(B_D) = B_D\,, \qquad \star(B^4) = B^4\,. \tag{A.20}$$

To find the star for the components, one can check that

$$\star(\sigma_i) = -\sigma_i \tag{A.21}$$

for the q-Pauli matrices, so that

$$\star(B^4) = B^4, \quad \star(B^i) = -B^i. \tag{A.22}$$

For M, this can also be verified using the characteristic equation for M in the spin λ irrep,

$$(M - q^{2\lambda})(M - q^{-2\lambda-2}) = 0\,. \tag{A.23}$$

[15] This is not essential, but it simplifies things.

Consistency of reality constraint with transformations. We verify that the above reality constraint is consistent with the algebra of vector rotations $B \to \pi(u_1)B\pi(Su_2)$ for $u \in U_q(su(2))^V$, which is preserved in the presence of a brane. We determine how $\star(B)$ transforms under vector rotations:

$$\begin{aligned}\star(B) \to\ & \pi(\hat{w})\hat{w}\ (\pi(u_1)B\pi(Su_2))^\dagger\hat{w}^{-1}\pi(\hat{w}^{-1}) = \pi(\hat{w})\pi(S((u_2)^*))\hat{w}B^\dagger\hat{w}^{-1}\pi((u_1)^*)\pi(\hat{w}^{-1})\\ &= \pi(\hat{w})\pi(S((u^*)_1))\pi(\hat{w}^{-1}) \star (B)\pi(\hat{w})\pi((u^*)_2)\pi(\hat{w}^{-1})\\ &= \pi(\theta((u^*)_1)) \star (B)\pi(S\theta((u^*)_2)) = \pi((\theta u^*)_1) \star (B)\pi(S(\theta u^*)_2)\\ &= \pi(u_1) \star (B)\pi(Su_2) \end{aligned} \tag{A.24}$$

(using $S\theta = \theta S^{-1}$) provided

$$u \in \mathcal{G}^V = \left\{u \in U_q(su(2))^V;\ \ u = \theta u^* = \hat{w}(Su^*)\hat{w}^{-1}\right\}. \tag{A.25}$$

Therefore $\star(B_D) = B_D$ is preserved under rotations $u \in \mathcal{G}^V$, which seems to be the appropriate "real" rotation group (algebra) compatible with (3.6). $\mathcal{G}^V$ is closed under addition and multiplication. A similar calculation shows consistency with the transformations (3.4).

One can in fact show consistency with the full rotation group $U_q(so(4))_{\mathcal{F}}$. This is most easily done in terms of coactions of the dual quantum group with generators s,t [10], where the rotations take the form $B \to sBt^{-1}$. Then one can show that $\tilde{B} \to t\tilde{B}s^{-1}$, hence $\star(t) = t^{-1}$ and $\star(s) = s^{-1}$ will guarantee that $\star(B)$ transforms as $\tilde{B}$.

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